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REMARK ON A PAPER BY B. SINGH ON CERTAIN NUMERICAL CHARACTERS OF SINGULARITIES

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In various papers related to resolution of singularities, Hironaka used successfully two different numerical characters of the local ring of a singular point on an algebraic variety. These are the local Hilbert functions and the character ν^* (see [1, 2]). Hironaka studied – as a key step to the goal of desingularization in characteristic zero – their behaviour under permissible monoidal transformations and proved that stability of the Hilbert functions is equivalent to stability of the ν^* ([2]). This theorem was the only result relating the Hilbert function and ν^*, until recently B. Singh [4] introduced in this journal the matrix ν^{**}, which contains ν^* as first row and which determines the Hilbert function (see Section 1 for definitions and remarks).

The main result of Singh states that under a permissible monoidal transformation with trivial residue field extension ν^* is stable if and only if ν^{**} is stable. The aim of this note is to show that Hironaka's paper [2] contains all tools to extend Singh's result to the case of an arbitrary residue field extension. So if ν^* is stable under a permissible monoidal transformation an improvement of the singularity cannot be measured by ν^{**}.

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1.

We start by briefly recalling the definitions (see [4]). Let $A = \bigoplus_{n \geq 0} A_n$ be a graded algebra and assume that A_0 is a field. Let M be a finitely generated graded A-module, $x_1, \ldots, x_r$ homogeneous elements of M. Then $x_1, \ldots, x_r$ is called a standard base for M, if

(a) $x_1, \ldots, x_r$ is a minimal system of generators for M and
(b) $\deg(x_1) \leq \cdots \leq \deg(x_r)$.

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The degrees $\deg(x_j)$ are independent of the standard base. Therefore it makes sense to define

$$\nu_j(M)=\begin{cases}\deg(x_j) & \text{if } j\le r,\\ \infty & \text{if } j>r\end{cases}$$

and

$$\nu_*(M)=(\nu_1(M),\nu_2(M),\ldots).$$

Now take a minimal free resolution

$$\to E_i\to E_{i-1}\to\cdots\to E_0\to M\to 0$$

of M by finitely generated, free graded A-modules. Since this resolution is unique up to isomorphism, we may define

$$\nu_{ij}(M)=\nu_j(E_i),\quad i\ge 0,\ j\ge 1,$$

and we put

$$\nu_{**}(M)=(\nu_{ij}(M)).$$

Next we consider a local ring (R,M) and an ideal J in R. We use the notation $\mathrm{gr}_M(R)$ for the associated graded ring of R, $\mathrm{in}_M(f)$ for the initial form of an element $f\in R$ $(f\neq 0)$ and $\mathrm{gr}_M(J,R)$ for the ideal in $\mathrm{gr}_M(R)$ generated by $\{\mathrm{in}_M(f)\mid f\in J, f\neq 0\}$.

A set of elements $f_1,\ldots,f_r\in J$ will be called a standard base for J, if their initial forms $\mathrm{in}_M(f_1),\ldots,\mathrm{in}_M(f_r)$ are a standard base for $\mathrm{gr}_M(J,R)$. Furthermore we define

$$\nu^*(J)=\nu_*(\mathrm{gr}_M(J,R))\quad\text{and}\quad \nu^{**}(J)=\nu_{**}(\mathrm{gr}_M(J,R)).$$

Finally, recall that the local Hilbert functions are given by

$$H_R^{(0)}(n)=1_R(M^n/M^{n+1})$$

and

$$H_R^{(d)}(n)=\sum_{k=0}^{n}H_R^{(d-1)}(k)\quad\text{if } d\ge 1.$$

Then B. Singh proved:

Proposition 1 [4, Thm. 2.1]. *If (R,M) is a regular ring of dimension e and J any ideal of R, then*

$$H_{R/J}^{(d)}(n)=\binom{n+e-1+d}{e-1+d}-\sum_{i=0}^{\infty}(-1)^i\sum_{j=1}^{\infty}\binom{n-\nu^{(ij)}(J)+e-1+d}{e-1+d}.$$

From now on we fix a regular local ring (R,M), an ideal J in R and a prime ideal $P\supset J$ which is permissible for J.[1] (R',M') will denote a monoidal transformation of R with center P and J' the strict transform of J in R'. The residue fields of R and R' will be denoted by k and k' respectively. Then we can prove the following

[1] This means that R/P is regular and $\mathrm{gr}_{\bar P}(\bar R)$ is flat over R/P, where $\bar R=R/J$ and $\bar P=P/J$.

Theorem. *The following conditions are equivalent:*

(i) $H^{(\delta)}_{R'/J'}(n) = H^{(0)}_{R/J}(n)$ *for all* n, *where* $\delta = \mathrm{tr.d.}_k\, k'$.

(ii) $\nu^*(J') = \nu^*(J)$.

(iii) $\nu^{**}(J') = \nu^{**}(J)$.

The implication (ii) ⇒ (iii) is the content of this note and will be proved in the next section. (iii) ⇒ (i) follows immediately from Proposition 1. Finally, (i) ⇒ (ii) is the easier part of Hironaka's proof of the equivalence of (i) and (ii) in [2]. The additional condition (iii) may be thought of as giving a different view of Hironaka's proof of (ii) ⇒ (i).

2.

From now on we will always assume $\nu^*(J') = \nu^*(J)$. In order to prove (iii), we have to introduce some more notation and we will recall some results of [2]. First we choose a regular system of parameters $(x_0, x_1, \ldots, x_r, y_1, \ldots, y_s)$ of R such that

(a) $P = (x_0, x_1, \ldots, x_r)R$ and

(b) $PR' = x_0 R'$.

We put $S = R'/MR'$. This can be viewed as a localisation of the polynomial ring $k[t_1, \ldots, t_r]$, where t_i is the residue of $x_i/x_0 \bmod MR'$.

Having fixed the regular system of parameters for R, we identify $\mathrm{gr}_M(R)$ with the polynomial ring

$$k[X, Y] = k[X_0, X_1, \ldots, X_r, Y_1, \ldots, Y_s]$$

and $\mathrm{gr}_P(R) \otimes_R k$ with the subring $k[X] = k[X_0, X_1, \ldots, X_r]$. We have a homomorphism

$$\alpha : k[X] \to S, \quad \alpha(X_0) = 1, \quad \alpha(X_i) = t_i, \; i = 1, \ldots, r,$$

obtained by 'dehomogenizing'. Since we assume $\nu^*(J') = \nu^*(J)$, we are interested in elements $f \in J$ such that

$$\nu_{M'}(f/x_0^d) = d, \quad \text{where } d = \nu_M(f).$$

A first step to obtain a standard base of J with this propety is to consider the graded subalgebra $U \subset k[X]$ whose term of degree d is

$$U_d = \{\varphi \in k[X]_d \mid \alpha(\varphi) \in N^d\},$$

where N is the maximal ideal of S. It can be shown that, if $\varphi \in U_d$ and $\varphi \neq 0$, then $\nu_N(\alpha(\varphi)) = d$. Most important for the results of [2] and hence for our proof is now the following description of U, which is given in [3]:

Proposition 2. *After a permutation of $x_1, \ldots, x_r$ (if necessary) we have*

$$U = k[\sigma_1, \ldots, \sigma_e],$$

where

$$\sigma_i = X_i^{q_i} + \sum_{j=i+1}^{r+1} c_{ij} X_j^{q_i}, \quad 1 \le i \le e, \quad X_{r+1} = X_0$$

and

$$1 \le q_1 \le \cdots \le q_e.$$

Furthermore

$$q_i = \begin{cases} 1 & \text{if } \operatorname{char}(k) = 0, \\ \text{power of } p & \text{if } \operatorname{char}(k) = p > 0. \end{cases}$$

Corollary. $\operatorname{gr}_M(R)$ *is flat over* U.

Proof. Let $B = k[\sigma_1, \ldots, \sigma_e, X_{e+1}, \ldots, X_r, X_0]$. Then $k[X_0, \ldots, X_r]$ is a free B-module with basis $\{X_1^{b_1} \cdots X_e^{b_e} \mid b_i < q_i\}$. Therefore $\dim B = \dim k[X] = r + 1$ and B is a polynomial ring over U.

Proposition 3. *There exists a standard base $f_1, \ldots, f_m$ of J with the following properties:*

(a) *If $d_i = \nu_M(f_i)$ and $g_i = f_i/x_0^{d_i}$, there exist $w_i \in M(g_1, \ldots, g_{i-1})R'$ such that $h_i = g_i - w_i$, $i = 1, \ldots, m$, is a standard base of J'.*

(b) *$\phi_i = \operatorname{in}_M(f_i)$ is an element of U for $1 \le i \le m$.*

(c) *Let I be the ideal in $\operatorname{gr}_N(S)$ generated by $\operatorname{in}_N(\alpha(\phi_1)), \ldots, \operatorname{in}_N(\alpha(\phi_m))$ and let $\beta : \operatorname{gr}_{M'}(R') \to \operatorname{gr}_N(S)$ be induced by the surjection $R' \to S$. Then*

$$I = \beta(\operatorname{gr}_{M'}(J', R'))$$

These facts follow from the considerations proving Lemmas 15, 18, and 20 of [2].

For the proof of the following three lemmas we will fix a standard base of J with (a) to (c) of Proposition 3.

Lemma 1. $\nu^{**}(J) = \nu_{**}(\operatorname{gr}_M(J, R) \cap U)$.

Proof. By (b) of Proposition 3 we have

$$\operatorname{gr}_M(J, R) = (\operatorname{gr}_M(J, R) \cap U) \operatorname{gr}_M(R),$$

and $\operatorname{gr}_M(R)$ is flat over U, which proves the assertion.

Lemma 2. $\nu_{**}(\operatorname{gr}_M(J, R) \cap U) = \nu_{**}(I)$

Proof. Following Hironaka [2] we introduce

$$S_0 = k[t_{e+1}, \ldots, t_r]_{N_0'},$$

where

$$N_0' = N \cap k[t_{e+1}, \ldots, t_r], \qquad N_0 = N_0' S_0.$$

By [2], Lemma 14, $S_0/N_0 = S/N$. Therefore there are $\beta_i \in S_0$, $1 \le i \le e$, such that the images of $z_i := t_i - \beta_i$, $1 \le i \le e$, form a regular system of parameters of S/N_0S, see [2, 14.1]. By Proposition 2 we have

$$\begin{aligned} \alpha(\sigma_i) &= t_i^{q_i} + \sum_{j=i+1}^{r} c_{ij} t_j^{q_i} + c_{i,r+1} \\ &= z_i^{q_i} + \sum_{j=i+1}^{e} c_{ij} z_j^{q_i} + \gamma_i, \quad \gamma_i \in S_0. \end{aligned}$$

Putting $\tau_i = \mathrm{in}_N \alpha(\sigma_i)$ it follows that

$$\tau_i = Z_i^{q_i} + \sum_{j=i+1}^{e} c_{ij} Z_j^{q_i} + \eta_i, \quad \eta_i \in \mathrm{gr}_{N_0}(S_0),$$

where $\mathrm{gr}_N(S)$ is identified with $\mathrm{gr}_{N_0}(S_0)[Z_1, \ldots, Z_e]$. Defining $\alpha^* : U \to \mathrm{gr}_N(S)$ by $\alpha^*(\sigma_i) = \tau_i$, $\alpha^*(U)$ can be identified with the subring $k[\tau_1, \ldots, \tau_e]$ of $\mathrm{gr}_N(S)$. Therefore we conclude as in the corollary to Proposition 2 that α^* is flat. But

$$\alpha^*(\mathrm{gr}_M(J, R) \cap U) \cdot \mathrm{gr}_N(S) = I.$$

This proves Lemma 2.

Lemma 3. $\nu_{**}(I) = \nu^{**}(J')$.

Proof. By definition we have $\nu^{**}(J') = \nu_{**}(\mathrm{gr}_{M'}(J', R'))$. Let $X_0' = \mathrm{in}_{M'}(x_0)$, $Y_j' = \mathrm{in}_{M'}(y_j)$ in $\mathrm{gr}_{M'}(R')$ $(1 \le j \le s)$. Then

$$\mathrm{gr}_N(S) = \mathrm{gr}_{M'}(R') / (X_0', Y_1', \ldots, Y_s')\, \mathrm{gr}_{M'}(R')$$

and $X_0', Y_1', \ldots, Y_s'$ is a regular sequence for the $\mathrm{gr}_{M'}(R')$-module

$$\mathrm{gr}_{M'}(R') / \mathrm{gr}_{M'}(J', R')$$

[2, Lemma 23]. Now it follows from [4, Cor. (3.2)] that

$$\nu^{**}(J') = \nu_{**}(\beta(\mathrm{gr}_{M'}(J', R'))),$$

where β is as in Proposition 3. Now the lemma follows from (c) of Proposition 3, and this also finishes the proof of our theorem.

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